RESEARCH PROJECT: PROBABILISTIC AND COMBINATORIAL METHODS IN HARMONIC ANALYSIS (PCMHA)

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Overview of the topics. The interplay between Harmonic Analysis and Potential Theory on one side, and Probability and Combinatorial Methods on the other, is in itself a vast, diverse, and evolving field. At its inception there were a few seminal ideas, which can be roughly grouped into three chapters.

On the one hand, there was the realization, going back at least to Zygmund in the 1930's, that cancellations occurring in trigonometric series are similar to, and intertwined with, those occurring in stochastic processes such as martingales. On the other hand, a by now well known vocabulary allows one to translate objects, phenomena, and results from the theory of Markov Processes to Potential Theory (exit distribution=harmonic measure; occupation time=Green's function; and so on). When probabilistic ideas are reduced to their "discrete bone", they reveal their combinatorial nature in the advanced theory no less than in the classical, elementary one. Last, combinatorial tools, sometimes old, sometimes developed for the application at hands, are ubiquitous in Harmonic Analysis and Potential Theory. For instance, Harmonic Analysis in homogeneous spaces á la Coifman and Weiss reduces many problems concerning singular integrals, maximal functions, and the theory of the related function spaces, to a discrete-combinatorial theory just slightly in disguise.

The research project PCMHA is centered around two more specific, yet rather vast, areas of investigation, and their possible connections.

- **(A) Applications of Stochastic Optimal Control Theory to Potential Theory and Harmonic Analysis (Bellman functions thechniques).**
- **(B) Potential Theory on Ahlfors-regular metric spaces poor of rectifiable curves.**

A few words on each of these topics are in order.

(A) The Bellman function technique vastly generalizes subharmonicity arguments, which have a long history in analysis and, in turn, extend convexity techniques in proving integral inequalities. It was Burkholder (see e.g. the foundational [Bu1984], or [Ba2010]) who first realized that sharp inequalities for some operators acting on martingales could be obtained by finding the exact solution of a problem in Optimal Stochastic Control (the "Bellman function" of the problem). Gundy and Varopoulos independently showed that many important singular integrals could be interpreted in terms of such martingale transformations [GV1979]. In 1997, Nazarov, Treil, and Volberg at the same time simplified (making it "dyadic") and extended Burkholder's theory to cover a number of new inequalities which were not believed to be within its range (see [NTV1997], but also the expository [NTV2001]), and many researchers were attracted in the field.

We have here (i) a very flexible method to obtain integral inequalities in a "dyadic setting", (ii) which can be "transferred" to a variety of settings (discrete and continuous: graphs, manifolds, Lie groups and the associated homogeneous spaces, ...) by means of old and new "averaging/randomizing techniques" (see e.g the method of generalized random grids in [H2010]).

(B) Motivated by the theory of quasi-conformal mapping between sub-Riemannian manifolds, in 1998 Heinonen and Koskela [HK1998] developed a powerful potential theory on metric-measure spaces of Ahlfors-regular spaces (i.e. spaces where the volume of the ball increases like a power of the radius) having enough rectifiable curves to make their definition of "very weak gradient" and the associated Poincaré inequality nontrivial. A few years before, the same authors [HK1995] and Hailasz [Ha1994], had developed different versions of potential theory on Ahlfors spaces; and it took some years to compare the three axiomatics. In 2014 [ARSW2014] a different version of potential theory on Ahlfors spaces was introduced, using potentials instead of gradients, in order to cover metric spaces with few, or no rectifiable curves. Many questions in the latter case can be answered by means of simpler dyadic models, and some techniques which allow their transplantation in the general context.

The project. PCMHA has at its basis some rather precise questions.

(A1) Recently [AHMV2021] a proof of the characterization of the $L^2 - L^2$ trace measures for the dyadic Hardy operator (i.e. for the dyadic version of a two-weight Hardy inequality) was obtained by means of a new "Bellman function", which also gives sharp information on the constants involved. Are there Bellman functions for the $L^p - L^q$ trace measures with $1 < p, q < \infty$? One expects the case $q < p$ to be more difficult.

The trace inequalities have the form $\|If\|_{L^q(\mu)} \leq C(\mu,\sigma)\|f\|_{L^p(\sigma)}$, where I is the dyadic Hardy operator and μ, σ are weights on the dyadic grid.

(A2) Is there a similar approach for the weighted, dyadic Moser-Trudinger inequalities?

(A3) Can be such inequalities be "transferred" to the nondyadic setting developed in [ARSW2010]?

(B1) If a Ahlfors space satisfies the hypothesis of [HK1998], are the potential theories developed in [ARSW2010] and [HK1998] the same? To be more precise, does a closed set E have comparable capacities with respect to the two theories?

(B2) A Ahlfors regular space can be viewed as the boundary of a graph, similarly to the way the unit circle can be viewed as the boundary of the hyperbolic plane. Can the potential theory "without curves" of [ARSW2010] be obtained by considering curves in the graph which "fills" the Ahlfors space?

Problems (A1) and (A2) are rather challenging, since they require finding new Bellman functions, hence a nontrivial amount of imagination and skill in using some deep analytic and/or stochastic tools. Question (B2) might provide a bridge to move results from potential theories "with many rectifiable curves" to settings where such curves are scarce. Even partial success on the objectives of the project would be an important progress in the two areas.

Problems (A1-A3) and (B1-B2) illustrate the content of the project, but of course this kind of investigation, by its nature, produces more questions, and more links and applications to other theories.

Timetable.

Months 1-3	(A) Preliminary study of the relevant literature and definition of the boundary value problem which has to be satisfied by the Belmman functions. (B) Study and comparison of different versions of potential theory in metric spaces in a series of seminars.
Months 4-6	(A) Solution of the boundary value problems for some of the inequalities. (B) Rigorous comparison of the theories in [HK1998] and [ARSW2014], with special emphasis on discrete-type analogues of Poincaré inequalities.
Months 7-10	(A) Solution of the remaining boundary value problems, transference of the results in a nondyadic setting. (B) Potential theory on the graph based on a metric space.
Months 11-12	Two/three articles with the results of the projects are being written.

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